MATH 2050C Lecture 25 (Apr 21)

Final Exam: May 5, 2022 12:30 PM - 2:30 PM

Topics to be covered: (refer to Bartle (4th Ed.))

- § 2.1 - 2.5 (except binary / decimal representations)
- § 3.1 - 3.5
- § 4.1 - 4.2
- § 5.1 - 5.4 (except approximation by step functions / polynomials)

REVIEW SESSION

Chapter 2 The Real Numbers

$\mathbb{R}$

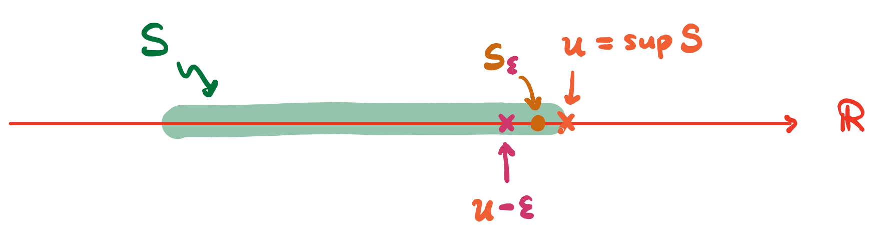
- **field** $\boxed{\S 2.1}$: $+, -, \times, \div$; field axioms
- **ordered** $\boxed{\S 2.1 - 2.2}$: "trichotomy" $a<b$, $a=b$, $a>b$
- **complete** $\boxed{\S 2.3 - 2.4}$: existence of $\sup S$, $\inf S$

Completeness Property $\boxed{\S 2.3}$: Every $\phi \neq S \subseteq \mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

Def$^{\underline{n}}$ $\boxed{\S 2.3}$:

$$u = \sup S \iff \begin{cases} u \geq s & \forall s \in S \\ \forall \varepsilon > 0, \ \exists s_\varepsilon \in S & \text{s.t. } u - \varepsilon < s_\varepsilon \end{cases}$$

Picture:



! Caution !
$\sup S \in S$
OR $\sup S \notin S$

Useful Inequalities $\boxed{\S 2.2}$: AM-GM ineq., (reversed) triangle ineq., Bernoulli's ineq.

Useful Facts $\boxed{\S 2.4}$:

- $\mathbb{N}$ is NOT bounded above (Archimedean Property)

(from completeness)

- Density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$
- Existence of $\sqrt{2}$

Intervals : $\boxed{\S 2.5}$

- characterization of intervals ("Connectedness")
- Nested Interval Property : ("compactness")

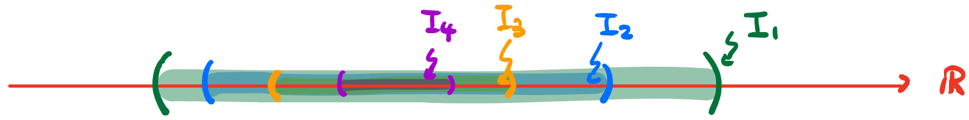
$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots\cdots \quad \Rightarrow \quad \bigcap_{n=1}^{\infty} I_n \neq \phi$$

closed and bounded intervals

$\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \phi$

$\bigcap_{n=1}^{\infty} [n, \infty) = \phi$

Picture:



Chapter 3 Sequences (and Series)

$\{x_n : n \in \mathbb{N}\}$ set $\neq$ seq. $(x_n) = (x_1, x_2, x_3, x_4, \ldots\ldots) : \mathbb{N} \to \mathbb{R}$

Def$^{\underline{n}}$: $\boxed{\S 3.1}$

$\lim (x_n) = L \iff \forall \varepsilon > 0, \exists K \in \mathbb{N}$ s.t. $|x_n - L| < \varepsilon \quad \forall n \geq K$

(K depends on ε)

$\boxed{\S 3.2}$

Limit Thm A : If $\lim(x_n)$ and $\lim(y_n)$ exist, then

- $\lim(x_n \pm y_n) = \lim(x_n) \pm \lim(y_n)$
- $\lim(x_n y_n) = \lim(x_n) \lim(y_n)$
- $\lim\left(\frac{x_n}{y_n}\right) = \frac{\lim(x_n)}{\lim(y_n)} \leftarrow$ Provided: $y_n \neq 0$, $\lim(y_n) \neq 0$

$(\frac{1}{n}) \to 0$

$\boxed{\S 3.2}$

Limit Thm B : If $\lim(x_n)$ and $\lim(y_n)$ exist, then

" $x_n \leq y_n \quad \forall n \in \mathbb{N} \Rightarrow \lim(x_n) \leq \lim(y_n)$ "

[! Caution ! Only get "$\leq$" even if $x_n < y_n \ \forall n \in \mathbb{N}$. E.g. $0 < \frac{1}{n}$]

$\boxed{\S 3.2}$

FACT : (x_n) convergent $\Longrightarrow$ (x_n) bounded

$\Longleftarrow$ + monotone

Monotone Convergence Thm $\boxed{\S 3.3}$

$(x_n) = ((-1)^n)$

$(x_n) = (\frac{1}{n})$

$(x_n) = (\frac{(-1)^n}{n})$

To show (x_n) divergent

(I) (x_n) unbounded §3.2

(II) $\exists$ two subseq of (x_n) §3.4

$(x_{n_k}) \longrightarrow L$

$\neq$

$(x_{m_k}) \longrightarrow L'$

To show (x_n) convergent

(I) $\varepsilon - K$ definition §3.1

(II) Limit thms §3.2

(III) Squeeze thm §3.2

do NOT need to know the limit

*(IV) Monotone Convergence Thm §3.3

*(V) Cauchy criteria §3.5

Def$^{\underline{n}}$: §3.5

(x_n) is Cauchy $\iff$ $\forall \varepsilon > 0$, $\exists H \in \mathbb{N}$ s.t. (depends on ε)

$|x_n - x_m| < \varepsilon \quad \forall \underline{n, m} \geq H$

no relation between them

Cauchy Criteria: §3.5 (x_n) convergent $\iff$ (x_n) Cauchy

"iff"

Bolzano-Weierstrass Thm: §3.4 Any bounded seq has a convergent subseq.

[! Caution ! May have different subseq's converging to different limits.

E.g. $((-1)^n)$ $\rightsquigarrow$ limsup & liminf]

Chapter 4 Limits (of functions)

Setup: $f: A \longrightarrow \mathbb{R}$, $c \in \mathbb{R}$ is a cluster pt of A

[! Caution ! Either $c \in A$ or $c \notin A$ is possible E.g.) $A = [0,1)$]

Def$^{\underline{n}}$: §4.1

$\lim_{x \to c} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0$ s.t. (depends on ε)

$|f(x) - L| < \varepsilon \quad \forall x \in \underline{A},\ \underbrace{0 < |x - c| < \delta}_{x \neq c}$

Sequential Criteria: §4.1

$\lim_{x \to c} f(x) = L$ (limit of function) $\iff$ $\forall$ seq. (x_n) in $\underbrace{A \setminus \{c\}}_{x_n \neq c}$ s.t. $\lim(x_n) = c$

$\lim(f(x_n)) = L$ (limit of seq.)

[FACT: Useful to show $\lim_{x \to c} f(x)$ does NOT exist. E.g.) $f(x) = \sin \frac{1}{x}$]

• Limit Thm A and B carries over from seq. to functions $\boxed{\S 4.2}$

Chapter 5 Continuous Functions

$\boxed{\S 5.1}$

<u>Def^{n}</u>: $f : A \to \mathbb{R}$ is continuous at $c \in A$ $\iff$ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - f(c)| < \varepsilon$ $\forall x \in A$, $|x - c| < \delta$

depends on ε (and c)

no $0<$

[! Caution ! Unlike $\lim_{x \to c} f(x)$, we NEED $c \in A$ here.]

<u>Sequential Criteria</u>: $\boxed{\S 5.1}$

$f : A \to \mathbb{R}$ is cts at a cluster pt. $c \in A$ $\iff$ $\lim(f(x_n)) = f(c)$ $\forall$ seq. (x_n) in A s.t. $\lim(x_n) = c$

[FACT: Useful to show Discontinuity at c. E.g.) $f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ 0 & , x \notin \mathbb{Q} \end{cases}$]

$\boxed{\S 5.2}$

<u>Facts</u>: f, g cts $\Rightarrow$ $f \pm g$, fg, f/g, $f \circ g$ new!

composition

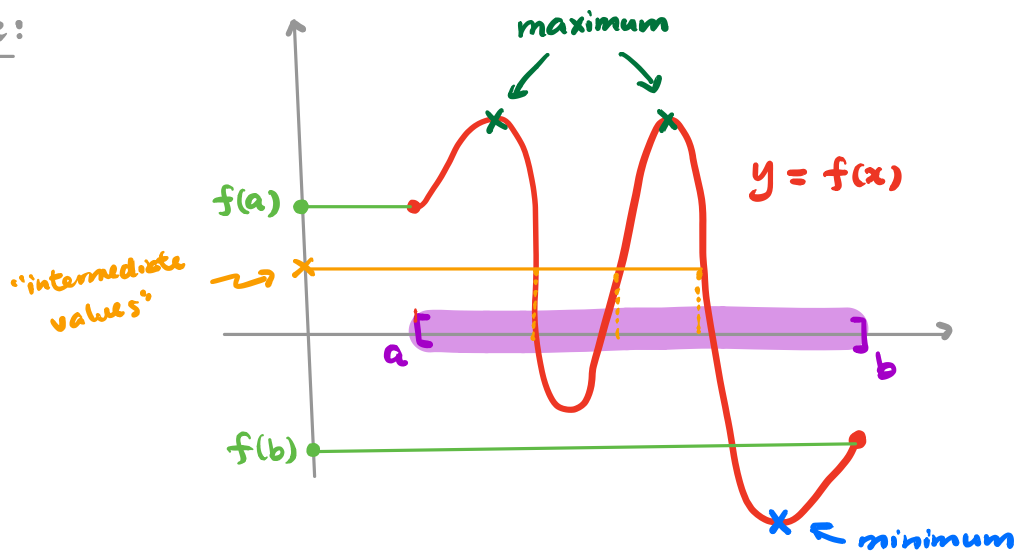
<u>Two Theorems for cts $f : [a, b] \to \mathbb{R}$</u> $\boxed{\S 5.3}$

closed + bdd

<u>Extreme Value Thm</u>: f achieves its absolute maximum and minimum.

<u>Intermediate Value Thm</u>: f achieves ALL intermediate values between $f(a)$ and $f(b)$.

<u>Picture</u>:



Def^{n}: $\boxed{\S 5.4}$

$f : A \to \mathbb{R}$ is uniformly cts (on A) $\iff$ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(u) - f(v)| < \varepsilon$ $\forall u, v \in A$, $|u - v| < \delta$

depends ONLY on ε, but NOT u, v

FACT: f unif. cts on A $\Longrightarrow$ f cts on A (i.e. at ALL $c \in A$)

$\not\Longleftarrow$ $\because$ δ may depend on $c \in A$

e.g.) $f(x) = x$ e.g. $f(x) = \frac{1}{x}$

Two Important Thm about uniform continuity $\boxed{\S 5.4}$

Uniform Continuity Thm:

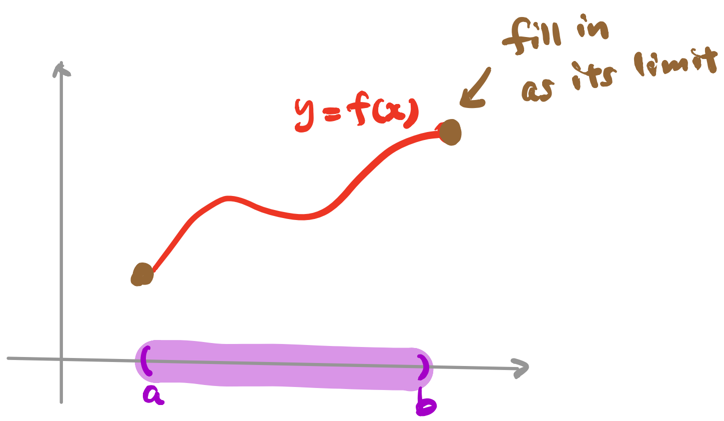
Any cts $f : [a, b] \longrightarrow \mathbb{R}$ is uniformly cts.

closed + bdd

Continuous Extension Thm:

Any uniformly cts $f : (a, b) \longrightarrow \mathbb{R}$ can be continuously extended to $[a, b]$.

Picture:



~ END OF REVIEW SESSION ~

Good Luck!